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An Application of Inversion to Wave Propagation

by

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1. Introduction.

The problem of propogation of radio waves in a homogeneous atmosphere has been studied theoretically by several investigators, beginning about 1909 with the work of Sommerfeld. A brief account of this work will be found in Stratton's Electromagnetic Theory, p.573 ff. In this work the effect of reflection from the earth and from the ionosphere has been treated. This problem can be said to be largely solved, at least for long wave-lengths.

In the case of an inhomogeneous non-standard atmosphere the problem of propagation offers great difficulties. Attempts have been made to approximate actual atmospheres, and the resulting problems have been solved, also approximately. One such method - that carried out by W. H. Furry - approximates some possible atmospheric condition by two layers, each with a constant refractive index. The method is theoretically suitable for any number of layers. In this study the layers are plane, and in order to obtain results applicable to a spherical earth surrounded by layers in the form of spherical shells, an earth flattening factor is used. This is, of course, a further approximation. Even the approximate solutions, for limited cases, have led to immense amounts of calculation in which progress has been slow.

The present paper represents an attempt to deal with the problem of propogation in a special case of an inhomogeneous atmosphere so as to obtain an exact solution. The problem is the following: We consider a spherical earth, which is finitely conducting and within which the index of refraction is constant, surrounded by a layer of finite depth in which the index of refraction varies inversely as the square of the distance from the earth's center. Outside this layer is an infinite layer in which the refractive index is constant. A source of radiation is situated in the finite layer. We are to find the electromagnetic field at any point.

We have restricted the layer with variable index of refraction to finite depth since it is clear that, while such a variation of the refraction index is possible in a finite layer, if the layer were extended infinitely far we would have the refractive index tending to 0, which is physically untrue.

The method of attack is the following. The wave equation in spherical coordinates satisfied by the Hertzian potential function in the case of a constant index of refraction is

$$\nabla^2 u + k^2 u = 0. \qquad (A)$$

A complete set of solutions of this equation is given in terms of Bessel and Hankel functions and the Legendre polynomials, in the case of symmetry about an axis. If we let $\rho = \frac{a^2}{r}$, where a = the radius of the earth, (this is the inversion in the sphere representing the earth's surface, with the earth's center as the center of inversion) equation (A) becomes

$$\nabla^2 U + \frac{k^2 a^2}{r^4} U = 0 , \qquad (B)$$

and it can be shown that the function

$$U(r,\theta,\varphi) = \frac{1}{r}\, u\left(\frac{a^2}{r}, \theta, \varphi\right)$$

satisfies this modified wave equation. Hence, from the complete set of solutions of (A) we are able immediately to write down a complete set of solutions of (B). Since (B) is the wave equation satisfied by the Hertzian potential function in the case of a refractive index which varies inversely as the square of r, we are in a position to solve the problem described above, provided the boundary conditions can be met.

The boundary conditions are derived from Maxwell's equations. In the case we are considering the quantity ε, usually called the dielectric "constant", is not constant but varies inversely as the fourth power of r. It is found that the boundary conditions in this case are essentially the same as for a constant ε. These conditions are that εu and $\frac{\partial (ru)}{\partial r}$ must be continuous across the bounding surfaces of the layer.

The solution of the problem is straightforward and only minor approximations have been used. The result is in the form of an infinite series of Bessel and Hankel functions, with coefficients found from four simultaneous linear equations. If the series converged rapidly the calculation of the necessary coefficients would not be an insurmountable obstacle. But, unfortunately, the series converges very slowly and the

work necessary to obtain useful results would be prohibitive. It must be remembered that results calling for a formidable amount of calculation have been encountered before, notably in the case of Watson's solutions of the propagation problem in a homogeneous atmosphere. The work of Eckersley* and that of van der Pol and Bremmer** was aimed at simplifications of these results so as to make the calculations reasonably simple. It is conceivably that the results obtained in this paper can also be brought within the range of practical computation. At present it does not seem that the case covered has sufficient application to warrant the attempt, at least not until other avenues have been explored.

This work has some relationship to that of W. H. Furry mentioned above, as may be seen from the following consideration. Furry makes use of the modified index of refraction, N, defined as follows:

$$N = \frac{\text{(index of refraction) x (distance to earth's center)}}{\text{same product taken at a special reference point}} \ .$$

In the case of two layers, each with a constant index of refraction, the modified index has the form

$$N = m_1 r, \qquad r < r_1,$$

$$N = m_2 r, \qquad r > r_1,$$

where r_1 is the value of r at the interface between the layers. This leads to the so-called bilinear M-curve made up of two segments with different slopes as shown in Fig. 1. With two layers, in the first of

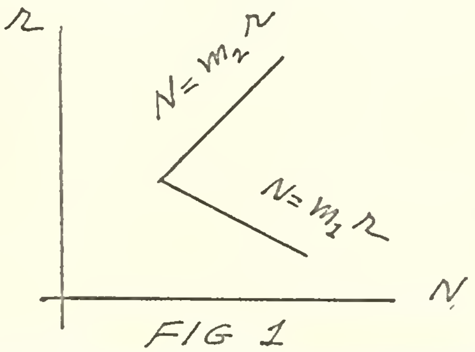

FIG 1

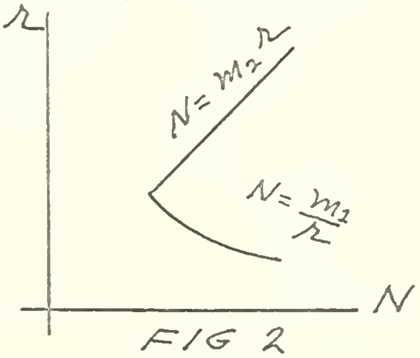

FIG 2

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* Eckersley and Millington, Royal Soc. Phil. Trans., v. 237, 1939, p 273; Phil. Mag., v.27, 1939, p 517.

** van der Pol and Bremmer, Phil. Mag., v. 24, 1937, p 141, p 825; v. 25, 1938, p 817.

which the index of refraction varies inversely as the square of r and in the second of which it is constant, we have for the modified index

$$N = \frac{m_1}{r}, \qquad r < r_1,$$

$$N = m_2 r, \qquad r > r_1.$$

The M-curve in this case is that shown in Fig. 2.

It should be emphasized that we are dealing with a spherical earth from the beginning and no earth flattening device is necessary. This study may be considered to be a first step in an attempt to solve the propogation problem for inhomogeneous atmospheres closely approximating true conditions. It seems best to follow up this work by the use of other expressions for the index of refraction that will leave the wave equation in some manageable form and will leave the boundary conditions essentially the same as those used here. An attractive possibility, which will be investigated, is that in which $\varepsilon = 1 + \frac{k}{r^{\alpha}}$, where r is the distance from the center of the earth. Under this assumption it will not be necessary to consider the atmosphere to be in layers; ε can vary continuously from r = radius of the earth to r = infinity, and as r tends to infinity ε tends to 1 as physical considerations show that it should. In this case the only boundary surface on which special conditions will have to be met will be the earth's surface.

2. Mathematical Formulation of the Problem.

Since we are interested in propagation in a layer in which ε is variable, we must examine the wave equation to be satisfied in this layer and also the boundary conditions that must be met.

Maxwell's equations are

$$\nabla \times E = -\frac{\partial B}{\partial t}, \quad (1) \qquad \nabla \times H = \frac{\partial D}{\partial t}, \quad (2)$$

$$\nabla \cdot B = 0, \quad (3) \qquad \nabla \cdot D = 0, \quad (4)$$

in which $D = \varepsilon E$ and $B = \mu H$. We assume μ to be constant but suppose that ε is variable from point to point. We shall introduce a Hertzian vector, Π, in the usual way.

Let

$$H = \nabla \times \frac{\partial(\varepsilon \Pi)}{\partial t}. \qquad (5)$$

This expression automatically satisfies (3). Substituting in (1),

$$\nabla \times E + \mu \nabla \times \frac{\partial^2(\varepsilon \Pi)}{\partial t^2} = 0,$$

or

$$\nabla \times \left(E + \mu\varepsilon \frac{\partial^2 \Pi}{\partial t^2} \right) = 0.$$

This equation will be satisfied if we put the quantity in parentheses equal to the gradient of a scalar function. Let

$$E + \mu\varepsilon \frac{\partial^2 \Pi}{\partial t^2} = \nabla \Psi , \qquad (6)$$

where Ψ is yet to be determined.

Substituting from (5) and (6) into (2), we have

$$\nabla \times (\nabla \times \varepsilon \Pi) = D = \varepsilon \nabla \Psi - \mu \varepsilon^2 \frac{\partial^2 \Pi}{\partial t^2} . \qquad (7)$$

Because of the nature of the propagation problem in which we are interested, we adopt a spherical coordinate system and assume that ε depends upon r only. For later purposes it will be sufficient to assume that Π has only an r-component, Π_r. Then

$$\nabla \times \varepsilon \Pi = \left(0, \frac{1}{r \sin\theta} \frac{\partial(\varepsilon \Pi_r)}{\partial \varphi}, -\frac{1}{r} \frac{\partial(\varepsilon \Pi_r)}{\partial \theta} \right), \qquad (8)$$

and

$$\nabla \times (\nabla \times \varepsilon \Pi) = \left(-\frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial(\varepsilon \Pi_r)}{\partial \theta}\right) - \frac{1}{r^2 \sin^2\theta} \frac{\partial^2(\varepsilon \Pi_r)}{\partial \varphi^2}, \right.$$

$$\left. \frac{1}{r} \frac{\partial^2(\varepsilon \Pi_r)}{\partial r\, \partial \theta}, \frac{1}{r \sin\theta} \frac{\partial(\varepsilon \Pi_r)}{\partial r\, \partial \varphi} \right). \qquad (9)$$

Since

$$\nabla \Psi = \left(\frac{\partial \Psi}{\partial r}, \frac{1}{r} \frac{\partial \Psi}{\partial \theta}, \frac{1}{r \sin\theta} \frac{\partial \Psi}{\partial \varphi} \right),$$

we find, using the θ- and φ- components as determined from (7) and (9),

$$\frac{\varepsilon}{r} \frac{\partial \Psi}{\partial \theta} = \frac{1}{r} \frac{\partial^2(\varepsilon \Pi_r)}{\partial r\, \partial \theta} , \qquad (10)$$

and

$$\frac{\varepsilon}{r \sin \theta} \frac{\partial \Psi}{\partial \varphi} = \frac{1}{r \sin \theta} \frac{\partial^2 (\varepsilon \Pi_r)}{\partial r \, \partial \varphi} \qquad (11)$$

Equations (10) and (11) are satisfied if we take

$$\Psi = \frac{1}{\varepsilon} \frac{\partial (\varepsilon \Pi_r)}{\partial r} . \qquad (12)$$

Putting this in the r-component as found from (7) and (9), we have

$$\varepsilon \frac{\partial}{\partial r}\left(\frac{1}{\varepsilon} \frac{\partial (\varepsilon \Pi_r)}{\partial r}\right) + \frac{\varepsilon}{r^2 \sin \theta} \frac{\partial}{\partial \theta}\left(\sin \theta \frac{\partial \Pi_r}{\partial \theta}\right) + \frac{\varepsilon}{r^2 \sin^2 \theta} \frac{\partial^2 \Pi_r}{\partial \varphi^2} - \mu \varepsilon^2 \frac{\partial^2 \Pi_r}{\partial t^2} = 0,$$

or

$$\frac{\partial}{\partial r}\left(\frac{1}{\varepsilon} \frac{\partial (\varepsilon \Pi_r)}{\partial r}\right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta}\left(\sin \theta \frac{\partial \Pi_r}{\partial \theta}\right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \Pi_r}{\partial \varphi^2} - \mu \varepsilon \frac{\partial^2 \Pi_r}{\partial t^2} = 0, \quad (13)$$

We now assume that Π_r depends on the time through a factor of the form $e^{-i\omega t}$ only, where ω is 2π times the frequency of the radio waves. We may then write.

$$\Pi_r = r u(r, \theta, \varphi) e^{-i\omega t}.$$

Putting this in (13) and dividing by r, we get the modified wave equation:

$$\frac{1}{r} \frac{\partial^2 (ru)}{\partial r^2} + \frac{\varepsilon'}{\varepsilon r} \frac{\partial (ru)}{\partial r} + u \cdot \frac{\varepsilon \cdot \varepsilon'' - (\varepsilon')^2}{\varepsilon^2} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta}\left(\sin \theta \frac{\partial u}{\partial \theta}\right)$$

$$+ \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \varphi^2} + \mu \varepsilon \omega^2 u = 0,$$

or

$$\nabla^2 u + \frac{\varepsilon'}{\varepsilon r} \frac{\partial (ru)}{\partial r} = u \cdot \frac{\varepsilon \cdot \varepsilon'' - (\varepsilon')^2}{\varepsilon^2} + \mu \varepsilon \omega^2 u = 0. \qquad (14)$$

In the case in which we are interested $u = \frac{e^{iKR}}{R}$ and $\varepsilon = \frac{k^2 a^4}{r^4}$.

In these expressions

r = distance from any point to the center of the earth,

a = radius of the earth,

$k = \dfrac{2\pi}{\lambda}$,

R = distance from the transmitter to any point in the medium

$= \sqrt{r^2 + b^2 - 2rb\cos\theta}$,

where b is the distance from the transmitter to the center of the earth and θ is the angle between b and r.

With these values, keeping in mind that ε differs but little from 1, that R and $\frac{\partial R}{\partial r}$ are small in comparison with r, we find that the second and third terms of equation (14) are roughly $\frac{1}{r}$ and $\frac{1}{r^2}$ times the magnitude of the first and last terms, respectively. We see, therefore, that we may assume that u must satisfy the wave equation in spherical coordinates,

$$\nabla^2 u + \mu\varepsilon\omega^2 u = 0. \qquad (15)$$

From the knowledge of u we can find E and H from (5) and (6). We have, suppressing the factor $e^{-i\omega t}$,

$$H_r = 0, \quad H_\theta = -\frac{i\omega\varepsilon}{\sin\theta}\,\frac{\partial u}{\partial\varphi}, \quad H_\varphi = -i\omega\varepsilon\,\frac{\partial u}{\partial\theta}, \qquad (16)$$

$$E_r = \mu\varepsilon\omega^2 ru + \frac{\partial}{\partial r}\left(\frac{1}{\varepsilon}\,\frac{\partial(\varepsilon ru)}{\partial r}\right., \quad E_\theta = \left.\frac{1}{\varepsilon r}\,\frac{\partial^2(\varepsilon ru)}{\partial r\,\partial\theta}\right),$$

$$E_\varphi = \frac{1}{\varepsilon r\sin\theta}\,\frac{\partial^2(\varepsilon ru)}{\partial r\,\partial\varphi}. \qquad (17)$$

If we assume axial symmetry, derivatives with respect to φ vanish. On the boundaries, therefore, we must satisfy the conditions that H_φ and E_θ are continuous. Since r is constant on the bounding surfaces, we may integrate H_φ and E_θ with respect to θ. Hence we find that εu and $\frac{1}{\varepsilon}\frac{\partial(\varepsilon ru)}{\partial r}$ must be continuous on the boundaries. The second condition requires that $\frac{\partial(ru)}{\partial r} + \frac{\varepsilon'}{\varepsilon} ru$ be continuous, and, just as in the case of equation (14), we can show that $\frac{\varepsilon'}{\varepsilon} ru$ may be neglected in comparison

with $\frac{\partial(ru)}{\partial r}$. We can therefore make use of the following boundary conditions, namely, that εu and $\frac{\partial(ru)}{\partial r}$ must be continuous.

3. Inversion of the Wave Equation.

Let $u(\rho, \theta, \varphi)$ satisfy the wave equation,

$$\nabla^2 u + k^2 u = 0. \tag{18}$$

It can be shown by direct substitution that

$$U(r, \theta, \varphi) = \frac{1}{r}\, u\left(\frac{a^2}{r}, \theta, \varphi\right) \tag{19}$$

satisfies the modified wave equation,

$$\nabla^2 U + \frac{k^2 a^2}{r^4} U = 0, \tag{20}$$

into which (18) is transformed by putting $\rho = \frac{a^2}{r}$.

If we put

$$\begin{aligned} \zeta_n(x) &= \sqrt{\frac{\pi}{2x}}\, H^{(1)}_{n+1/2}(x), \\ \psi_n(x) &= \sqrt{\frac{\pi}{2x}}\, J_{n+1/2}(x), \end{aligned} \tag{21}$$

in which $H^{(1)}$ is the Hankel function of the first kind and J is the Bessel function of the first kind, the functions $\zeta_n(k\rho)P_n(\cos\theta)$ and $\psi_n(k\rho)\,P_n(\cos\theta)$ form a complete set of solutions of equation (18) in the case of axial symmetry. In these functions, the quantities $P_n(\cos\theta)$ are the Legendre polynomials with argument = $\cos\theta$. It follows that the functions

$$\frac{1}{r}\,\zeta_n\left(\frac{ka^2}{r}\right)P_n(\cos\theta), \quad \frac{1}{r}\,\psi_n\left(\frac{ka^2}{r}\right)P_n(\cos\theta), \tag{22}$$

form a complete set of solutions of equation (20), in the case of axial symmetry.

4. Formal Solution of the Proposed Propagation Problem.

We now consider the propagation problem stated in the introduction. The earth is assumed to be a homogeneous imperfectly conducting

sphere of radius a. In a layer, I, bounded by the earth and an outer spherical surface of radius d, the refractive index is assumed to vary inversely as the square of r, the distance to the earth's center. Outside this layer the refractive index is a constant equal to that for free space. We assume that a transmitter, behaving like a Hertzian dipole of angular velocity ω, is situated in I, at distance b from the center of the earth. We shall evaluate the field at any point P with spherical coordinates (r, θ, φ).

In order to obtain the solution of this problem, which is associated with equation (20), we first consider the propagation problem associated with equation (18). This is the case in which the refractive index is constant in the layer containing the transmitter. The field at any point in this layer will be the resultant of the field of the transmitter and the field produced by the earth and the outer region. We therefore put $u = u_t + u_a$, in which u_t is the Hertzian potential for the transmitter, and u_a is the Hertzian potential for the disturbed field. Letting $u_t = \frac{e^{ikR}}{ikR}$, where R represents the distance from the transmitter, we have *

$$u_t = \frac{e^{ikR}}{irR} = \sum_0^{\infty} (2n + 1)\zeta_n(kb)\psi_n(kr)\, P_n(\cos\theta), \quad r < b, \qquad (23a)$$

$$= \sum_0^{\infty} (2n + 1)\zeta_n(kr)\psi_n(kb)\, P_n(\cos\theta), \quad r > b, \qquad (23b)$$

where ζ_n and ψ_n have the meanings assigned to them in (21) and P_n are the Legendre polynomials. We may also write

$$u_a = \sum_0^{\infty} (2n + 1)\left[b_n \psi_n(kr) + c_n \zeta_n(kr)\right] P_n(\cos\theta), \qquad (24)$$

in which b_n and c_n have to be determined from the boundary conditions. This form is taken since the function u_a results from the superposition of waves reflected from the upper boundary and those reflected from the lower boundary. The first of these, which are incoming waves, are seen from (23a) to correspond to terms of the form $b_n \psi_n(kr)$. The waves

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*Stratton, J.A. : Electromagnetic Theory, p 414

reflected from the lower boundary are out-going waves, and hence, from (23b), correspond to terms of the form $c_n \zeta_n(kr)$.

Returning to the case in which the refractive index varies inversely as the square of r in layer I, we can also write the Hertzian function in this layer in the form

$$u = u_t + u_a. \qquad (25)$$

By applying the inversion to equations (23a), (23b) and (24), we find

$$u_t = \frac{b}{r} \sum_0^\infty (2n+1)\, \psi_n\left(\frac{ka^2}{r}\right) \zeta_n\left(\frac{ka^2}{b}\right) P_n(\cos\theta), \qquad r > b, \qquad (26a)$$

$$= \frac{b}{r} \sum_0^\infty (2n+1)\, \psi_n\left(\frac{ka^2}{b}\right) \zeta_n\left(\frac{ka^2}{r}\right) P_n(\cos\theta), \qquad r < b, \qquad (26b)$$

$$u_a = \frac{b}{r} \sum_0^\infty (2n+1) \left[b_n \psi_n\left(\frac{ka^2}{r}\right) + c_n \zeta_n\left(\frac{ka^2}{r}\right) \right] P_n(\cos\theta). \qquad (27)$$

In the outermost region, since the refractive index is constant and the waves are out-going only, the Hertzian function is

$$u_o = \sum_0^\infty (2n+1)\, a_n \zeta_n(k_o r)\, P_n(\cos\theta). \qquad r > d. \qquad (28)$$

In the earth, also, the refractive index is constant, and the waves are in-coming only. Hence we have for the Hertzian function inside the earth,

$$u_i = \sum_0^\infty (2n+1)\, d_n \psi_n(k_1 r)\, P_n(\cos\theta). \qquad r < a. \qquad (29)$$

We now apply the boundary conditions at r = d. Since εu must be continuous, the sum of (26a) and (27) must be equal to (28) when r is replaced by d. Writing this equality and dividing by $\frac{k^2 a^4}{d^4}$, we have

$$\frac{k_o^2}{K^2} a_n \zeta_n(k_o d) = \frac{b}{d} \left[b_n \psi_n\left(\frac{ka^2}{d}\right) + c_n \zeta_n\left(\frac{ka^2}{d}\right) + \psi_n\left(\frac{ka^2}{d}\right) \zeta_n\left(\frac{ka^2}{b}\right) \right] \qquad (30)$$

where $K = \frac{km^2}{d^2}$. Similarly, since $\frac{\partial (ru)}{\partial r}$ is continuous, we multiply (26a) and (27) by r, differentiate, replace r by d and add. This result must be equal to that obtained by multiplying (28) by r, differentiating, and setting r = d. Thus

$$k_o d \ a_n \zeta_n'(k_o d) = - \frac{bkm^2}{d^2} \Big[b_n \psi_n'(\frac{km^2}{d}) + c_n \zeta_n'(\frac{km^2}{d}) + \psi_n'(\frac{km^2}{d}) \zeta_n(\frac{km^2}{b}) \Big] . \qquad (31)$$

In obtaining the left-hand member of (31) we have used the approximation

$$\frac{d}{dr} \left[r \zeta_n(k_o r) \right]_{r=d} = k_o d \ \zeta_n'(k_o d) + \zeta_n(k_o d)$$

$$\sim k_o d \ \zeta_n'(k_o d),$$

since the term neglected is small in comparison with the one retained.

We apply the boundary conditions at r = a in a like manner. Since εu is continuous, the sum of (26b) and (27) must be equal to (29) when r is replaced by a. Writing this equality and dividing by k^2, we have

$$\frac{k_1^2}{k^2} \ d_n \psi_n(k_1 a) = \frac{b}{a} \left[b_n \psi_n(ka) + c_n \zeta_n(ka) + \psi_n(\frac{km^2}{b}) \zeta_n(ka) \right] \quad 32.$$

Since $\frac{\partial (ru)}{\partial r}$ is continuous, we multiply (26b) and (27) by r, differentiate, set r = a, and add. The result must be equal to that obtained by differentiating (29) and setting r = a. Thus

$$ak_1 d_n \zeta_n'(k_1 a) = - bk \Big[b_n \psi_n'(ka) + c_n \zeta_n'(ka) + \psi_n(\frac{ka^2}{b}) \zeta_n'(ka) \ . \qquad (33)$$

Equations(30) - (33) form a system of four equations, linear in a_n, b_n, c_n and d_n. Solving for these quantities by determinants,

we obtain

$$a_n = \frac{1}{\Delta}\begin{vmatrix} \psi_n(\frac{ka^2}{d}) & \zeta_n(\frac{ka^2}{d}) & -\psi_n(\frac{ka^2}{d})\zeta_n(\frac{ka^2}{b}) & 0 \\ \psi'_n(\frac{ka^2}{d}) & \zeta'_n(\frac{ka^2}{d}) & -\psi'_n(\frac{ka^2}{d})\zeta_n(\frac{ka^2}{b}) & 0 \\ \psi_n(ka) & \zeta_n(ka) & -\psi_n(\frac{ka^2}{b})\zeta_n(ka) & \frac{-ak_1^2}{bk^2}\psi_n(k_1a) \\ \psi'_n(ka) & \zeta'_n(ka) & -\psi_n(\frac{ka^2}{b})\zeta'_n(ka) & \frac{ak_1}{bk}\psi'_n(k_1a) \end{vmatrix}$$

$$b_n = \frac{1}{\Delta}\begin{vmatrix} -\psi_n(\frac{ka^2}{d})\zeta_n(\frac{ka^2}{b}) & \zeta_n(\frac{ka^2}{d}) & \frac{-dk_o^2}{bK^2}\zeta_n(k_od) & 0 \\ -\psi'_n(\frac{ka^2}{d})\zeta_n(\frac{ka^2}{b}) & \zeta'_n(\frac{ka^2}{d}) & \frac{k_od}{bK}\zeta'_n(k_od) & 0 \\ -\psi_n(\frac{ka^2}{b})\zeta_n(ka) & \zeta_n(ka) & 0 & \frac{-ak_1^2}{bk^2}\psi_n(k_1a) \\ -\psi_n(\frac{ka^2}{b})\zeta'_n(ka) & \zeta'_n(ka) & 0 & \frac{ak_1}{bk}\psi'_n(k_1a) \end{vmatrix}$$

$$c_n = \frac{1}{\Delta}\begin{vmatrix} \psi_n(\frac{ka^2}{d}) & -\psi_n(\frac{ka^2}{d})\zeta_n(\frac{ka^2}{b}) & -\frac{dk_o^2}{bk^2}\zeta_n(k_od) & 0 \\ \psi'_n(\frac{ka^2}{d}) & -\psi'_n(\frac{ka^2}{d})\zeta_n(\frac{ka^2}{b}) & \frac{k_od}{bK}\zeta'_n(k_od) & 0 \\ \psi_n(ka) & -\psi_n(\frac{ka^2}{d})\zeta_n(ka) & 0 & \frac{-ak_1^2}{bk^2}\psi_n(k_1a) \\ \psi'_n(ka) & -\psi_n(\frac{ka^2}{b})\zeta'_n(ka) & 0 & \frac{ak_1}{bk}\psi'_n(k_1a) \end{vmatrix}$$

$$d_n = \frac{1}{\Delta}\begin{vmatrix} \psi_n(\frac{ka^2}{d}) & \zeta_n(\frac{ka^2}{d}) & \frac{-dk_o^2}{bK^2}\,\zeta_n(k_o d) & -\psi_n(\frac{ka^2}{d})\,\zeta_n(\frac{ka^2}{b}) \\ \psi'_n(\frac{ka^2}{d}) & \zeta'_n(\frac{ka^2}{d}) & \frac{k_o d}{bK}\,\zeta'_n(k_o d) & -\psi'_n(\frac{ka^2}{d})\,\zeta_n(\frac{ka^2}{b}) \\ \psi_n(ka) & \zeta_n(ka) & 0 & -\psi_n(\frac{ka^2}{b})\,\zeta_n(ka) \\ \psi'_n(ka) & \zeta'_n(ka) & 0 & -\psi_n(\frac{ka^2}{b})\,\zeta'_n(ka) \end{vmatrix},$$

in which

$$\Delta = \begin{vmatrix} \psi_n(\frac{ka^2}{d}) & \zeta_n(\frac{ka^2}{d}) & \frac{-d\,k_o^2}{bK^2}\,\zeta_n(k_o d) & 0 \\ \psi'_n(\frac{ka^2}{d} & \zeta'_n(\frac{ka^2}{d}) & \frac{k_o d}{bK}\,\zeta'_n(k_o d) & 0 \\ \psi_n(ka) & \zeta_n(ka) & 0 & \frac{-ak_1^2}{bk^2}\,\psi_n(k_1 a) \\ \psi'_n(ka) & \zeta'_n(ka) & 0 & \frac{a\,k_1}{bk}\,\psi'_n(k_1 a) \end{vmatrix}$$

If these values of b_n and c_n are inserted in (26a), or (26b), and 27, and the resulting expressions for u_t and u_a are added, we have the Hertzian potential function at any point in the layer of variable index. Similarly inserting the values of a_n in (28), or d_n in (29), we have the expression of the Hertzian function at any point in the outer region, or any point inside the earth. It is not hard to see that these series do not converge rapidly. For large values of the argument the functions ψ_n and ζ_n are closely approximated by their asymptotic expressions, and these values change slowly with n. Hence succeeding values, both of the coefficients and of the variable factors in the series, change slowly.

It would therefore be necessary to use a large number of terms to obtain a good approximation to u. The computation of the coefficients of these terms would clearly entail a vast amount of labor, which does not seem justified in view of the limited application of the results.

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